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Unitarity of supersymmetric $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ and no-ghost theorem for fermionic strings in $AdS_3 \times \mathcal{N}$

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Abstract: The unitarity of the NS supersymmetric coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ is studied for the discrete representations. The results are applied to the proof of the no-ghost theorem for fermionic strings in $AdS_3 \times \mathcal{N}$ in the NS sector. A no-ghost theorem is proved for states in flowed discrete representations.

Keywords: Superstrings and Heterotic Strings, Superstring Vacua, Conformal Field Models in String Theory, Conformal and W Symmetry.

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Contents

1. Introduction

The purpose of this work is to study the unitarity of the supersymmetric euclidean $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset, which is the simplest example of a nontrivial noncompact Kazama-Suzuki model [1]. The coset is with respect to the timelike $\mathrm{U}(1)$ current of $\mathrm{SL}(2,\mathbb{R})$. Our result is a supersymmetric generalization of similar results for the bosonic coset [2], and is relevant for several models. We will use it here to fill a gap in the proof of the no-ghost theorem [3] for fermionic strings in $AdS_3 \times N$. We will also extend the proof to fermionic string excitations belonging to the flowed sectors introduced in [4].

The bosonic coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ is a non-linear sigma model whose target space geometry to first order in $1/k$ is that of a 2D black hole [6, 7]. In its supersymmetric version, the 2D black hole metric has been shown to describe the *exact* conformal background up to four loops [8], and has been argued to be exact to all orders [9]. Generalizing similar results for the bosonic case, the supersymmetric coset has been argued to be equivalent to certain matrix models [10] and to $N = 2$ Liouville theory [11, 12], and is also relevant in relation to Little String Theories [11].

The unitarity challenges presented by AdS_3 are very old [13]–[18]. A proof of the no-ghost theorem for bosonic strings on AdS_3 was given in [3], based on [5]. A different approach to unitarity for strings in AdS_3 has been advocated in [19].

The relation between the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset and string unitarity in $AdS_3 \times \mathcal{N}$ comes from the fact that the proof of the no-ghost theorem [5, 3] shows that physical states built upon $\mathrm{SL}(2,\mathbb{R})$ representations lie in the coset, modulo spurious states.[1] This was

[1]As shown in section 4, this statement does not hold for states with J_0^3 eigenvalue 0, which should be treated separately.

shown in [3] both for bosonic and fermionic strings. So string unitarity necessitates coset unitarity. The latter was proved in [2] for the bosonic coset, and we will prove it for the supersymmetric case in the discrete representations. It is worth noticing that both string and coset unitarity rely on truncating the spin j of the discrete representations. This restriction of j is rather ubiquitous, and we discuss it in section 2.

The plan of this work is as follows. In section 2 we review the basics of string quantization in AdS_3. In section 3 we prove the unitarity of the coset. In section 4 we review the no-ghost theorem for unflowed string states in the discrete series [3] and extend it to the flowed sectors. Section 5 contains the conclusions.

In this work we will only deal with the Neveu-Schwarz sector. The Ramond sector can be treated along the same lines.

2. Fermionic strings in AdS_3 and spectral flow

The worldsheet theory is composed by a supersymmetric $\mathrm{SL}(2,\mathbb{R})$ WZW model at level k with central charge

$$c_{\mathrm{SL}(2,\mathbb{R})} = 3 + \frac{6}{k} + \frac{3}{2} \tag{2.1}$$

and a supersymmetric unitary theory $\mathcal{N}$ with central charge $c_{\mathcal{N}} = 15 - c_{\mathrm{SL}(2,\mathbb{R})}$.

The current algebra of supersymmetric $\widehat{\mathrm{SL}}(2,\mathbb{R})$ at level k is generated by six currents $J^{3,\pm}, \psi^{3,\pm}$, whose modes satisfy

$$\begin{aligned}
[J^3_n, J^3_m] &= -\frac{k}{2} n \delta_{n+m,0} \\
[J^3_n, J^\pm_m] &= \pm J^\pm_{n+m} \\
[J^+_n, J^-_m] &= -2J^3_{n+m} + kn\delta_{n+m,0} \\
[J^3_n, \psi^\pm_m] &= \pm \psi^\pm_{n+m} \\
[J^\pm_n, \psi^\mp_m] &= \mp 2\psi^3_{n+m} \\
[J^\pm_n, \psi^3_m] &= \mp \psi^\pm_{n+m} \\
\{\psi^3_n, \psi^3_m\} &= -\frac{k}{2}\delta_{n+m,0} \\
\{\psi^+_n, \psi^-_m\} &= k\delta_{n+m,0} \, ,
\end{aligned} \tag{2.2}$$

with all other (anti)commutators vanishing, and the same for the antiholomorphic sector. The modding of J^a_n is integer and that of ψ^a_n half-integer (Neveu-Schwarz) or integer (Ramond). As usual, purely bosonic currents can be defined by

$$\begin{aligned}
j^a &= J^a - \hat{J}^a \\
\hat{J}^a &= -\frac{i}{k} f^a_{\ bc} \psi^b \psi^c \, .
\end{aligned} \tag{2.3}$$

The j^a form a $\widehat{\mathrm{SL}}(2,\mathbb{R})$ bosonic algebra at level $k+2$. The $\hat{J}^a$ and the ψ^a form a supersymmetric $\widehat{\mathrm{SL}}(2,\mathbb{R})$ algebra at level -2 which commutes with j^a. The spectrum of the theory is the direct product of the Hilbert spaces of both theories.

The Sugawara stress tensor and the supercurrent are

$$\begin{aligned} T &= \frac{1}{2k}[j^+j^- + j^-j^+] - \frac{1}{k}j^3j^3 - \frac{1}{2k}[\psi^+\partial\psi^- + \psi^-\partial\psi^+] + \frac{1}{k}\psi^3\partial\psi^3 \\ G &= \frac{1}{k}[\psi^+j^- + \psi^-j^+] - \frac{2}{k}\psi^3j^3 - \frac{2}{k^2}\psi^+\psi^-\psi^3 \end{aligned} \tag{2.4}$$

In particular the zero mode of T is

$$\begin{aligned} L_0 = &\frac{1}{k}\left[\frac{1}{2}(j_0^+j_0^- + j_0^-j_0^+) - j_0^3j_0^3 + \sum_{m=1}^{\infty} j_{-m}^+j_m^- + j_{-m}^-j_m^+ - 2j_{-m}^3j_m^3\right] + \\ &+\frac{1}{k}\sum_{m=\frac{1}{2}}^{\infty} m\left(\psi_{-m}^+\psi_m^- + \psi_{-m}^-\psi_m^+ - 2\psi_{-m}^3\psi_m^3\right). \end{aligned} \tag{2.5}$$

The highest weight representations of (2.2) are built as a direct product of representations of j^a and ψ^a. For the ψ^a currents, we have the usual representations for free fermions, for both Neveu-Schwarz and Ramond sectors. For the NS sector, in which we will be interested, we have a vacuum $|0\rangle$ annihilated by $\psi^a_{n>0}$ upon which the $\psi^a_{n<0}$ states act.

For j^a, we start from unitary representations $|j;t\rangle$ of the SL(2, $\mathbb{R}$) Lie algebra $j_0^{3,\pm}$. The representations are characterized by the eigenvalues $-j(j-1)$ of the Casimir operator $\frac{1}{2}(j_0^+j_0^- + j_0^-j_0^+) - (j_0^3)^2$. The states within each representation are labeled by the eigenvalue t of j_0^3. The states $|j;t\rangle$ are annihilated by $j^{3,\pm}_{n>0}$, and the Fock space of states is built by acting on them with $j^{3,\pm}_{n<0}$.

The unitary representations of the SL(2, $\mathbb{R}$) Lie algebra $j_0^{3,\pm}$ appearing in the spectrum of strings moving in an $AdS_3 \times \mathcal{N}$ background [4] are:

1. **Highest weight discrete representations**

   $$\mathcal{D}_j^+ = \{|j;t\rangle : t = j, j+1, j+2, \ldots\}$$

   where $|j;j\rangle$ is annihilated by j_0^- and j is a real number such that $1/2 < j < (k+1)/2$.

2. **Lowest weight discrete representations**

   $$\mathcal{D}_j^- = \{|j;t\rangle : t = -j, -j-1, -j-2, \ldots\}$$

   where $|j;-j\rangle$ is annihilated by j_0^+ and j is a real number such that $1/2 < j < (k+1)/2$.

3. **Continuous representations**

   $$\mathcal{C}_j = \{|j,\alpha;t\rangle : t = \alpha, \alpha\pm 1, \alpha\pm 2, \ldots\}$$

   where $0 \le \alpha < 1$ and $j = 1/2 + \mathrm{i}z$, where z is a real number.

The bounds for j appearing in $\mathcal{D}_j^\pm$ can be understood in terms of consistency conditions for the primary states. The lower bound, $1/2 < j$, is necessary for the normalizability of the primary states when their norm is interpreted as the $\mathcal{L}^2$ inner product of functions in the $\mathrm{SL}(2,\mathbb{R})$ group manifold [7]. As for the upper bound, $j < (k+1)/2$, it was noted in [11] that it is necessary for the unitarity of the primary states, when their norm is interpreted as the two-point function of the vertex operators creating them from the vacuum.

Moreover, adopting either the upper or the lower bound for j, the other one appears when imposing the $w = \pm 1$ spectral flow (see below) to be a symmetry of the spectrum. Finally, the compelling evidence for the correctness of these bounds on j comes from the fact that only this range of j appears in the spectrum of the thermal partition function of the model, computed by path integral techniques in [4].

Now, given any integer w, the algebra (2.2) is preserved by the spectral flow $J_n^{3,\pm} \to \tilde{J}_n^{3,\pm}$, $\psi_n^{3,\pm} \to \tilde{\psi}_n^{3,\pm}$ defined by

$$\begin{aligned} \tilde{J}_n^3 &= J_n^3 - \frac{k}{2} w \delta_{n,0} \\ \tilde{J}_n^\pm &= J_{n\pm w}^\pm \\ \tilde{\psi}_n^3 &= \psi_n^3 \\ \tilde{\psi}_n^\pm &= \psi_{n\pm w}^\pm \,. \end{aligned} \tag{2.6}$$

For $w = \pm 1$, the symmetry (2.6) maps the $\mathcal{D}_j^\mp$ representation into $\mathcal{D}_{k/2-j}^\pm$. But for generic w, it was shown in [4] that this symmetry maps regular $\mathrm{SL}(2,\mathbb{R})$ representations to new representations which must be included in a consistent quantization of strings in $AdS_3 \times \mathcal{N}$.

Note that the spectral flow was also performed on the fermions for consistency of the algebra. The implications of this simultaneous flow for the boundary CFT theory have been discussed in [22]. But the new sectors can be obtained by performing the flow only on the purely bosonic sector j^a. For the free fermions, it is easily seen to be a rearrangement of the spectrum. In fact, for $w > 0$ a flowed NS vacuum can be defined by

$$|\tilde{0}\rangle = \psi^-_{-\frac{1}{2}} \psi^-_{-\frac{3}{2}} \cdots \psi^-_{-|w|+\frac{1}{2}} |0\rangle \,,$$

while for $w < 0$ the ψ_n^- should be replaced by ψ_n^+. In both cases the new vacuum satisfies

$$\tilde{\psi}_n^{\pm,3} |\tilde{0}\rangle = 0, \qquad n > 0,$$

The flow (2.6) maps the modes of T and G to

$$\begin{aligned} \tilde{L}_n &= L_n + w J_n^3 - \frac{k}{4} w^2 \delta_{n,0} \\ \tilde{G}_r &= G_r + w \psi_r^3 \,. \end{aligned} \tag{2.7}$$

Physical states will satisfy, in the -1 picture [23],

$$\left(L_{n\geq 0} - \frac{1}{2} \delta_{n,0} \right) |x\rangle = G_{n>0} |x\rangle = 0 \tag{2.8}$$

The way to include the flowed representations is to impose the physicality conditions (2.8), with *unflowed* L_n and G_n, on $\tilde{j}^a$ descendants built upon $|\tilde{j}, \tilde{t}\rangle$ representations.

For generic w, the mass shell condition can be expressed as

$$-\frac{\tilde{j}(\tilde{j}-1)}{k} - wm + \frac{kw^2}{4} + N + h - \frac{1}{2} = 0 \tag{2.9}$$

where h a highest weight of the inner $\mathcal{N}$ theory and the level N is a half-integer number.

Since the spin in the continuous representation of $\mathrm{SL}(2,\mathbb{R})$ is $j = 1/2 + \mathrm{i}z$ with z real, the mass shell condition for unflowed strings is

$$\frac{1/4 + z^2}{k} + N + h - \frac{1}{2} = 0 \tag{2.10}$$

which can only be satisfied for $N = 0$. So string states in this sector will always be unitary because the zero mode representation is. This does not hold for flowed states in $\mathcal{C}_j$, which should be treated separately (see section 5).

3. Unitarity of supersymmetric $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$

The states in the $\mathrm{SL}(2,\mathbb{R})$ module can in general have negative norm because the currents J^3, ψ^3 are timelike. The states of the coset are states $|x\rangle$ such that $J^3_{n>0}|x\rangle = \psi^3_{n>0}|x\rangle = 0$ In this section we will show that the coset states built upon the $\mathcal{D}^{\pm}_j$ representations of the zero modes, have positive norm provided $k > 2$ and $j < k/2 + 1$. The proof is a generalization to the supersymmetric case of [2] for the bosonic coset.

The no-ghost theorem requires unitarity of states annihilated by $J^3_{n>0}$ and $\psi^3_{n>0}$. But if we were interested instead in proving the unitarity of states annihilated by $j^3_{n>0}$ and $\psi^3_{n>0}$ the result would follow immediately from that of the bosonic case, due to the direct product structure. A simple example of a state annihilated by $J^3_{n>0}$ and $\psi^3_{n>0}$ but not by $j^3_{n>0}$ is given by

$$|x\rangle = J^3_{-1}|j,t\rangle + \frac{1}{2}\psi^+_{-1/2}\psi^-_{-1/2}|j,t\rangle\,. \tag{3.1}$$

Moreover, only the condition imposed with $J^3_{n>0}$ is consistent with supersymmetry in the sense that if we require of our state to be simultaneously a primary of the supercurrent G, no further restrictions arise. If the conditions are imposed with $j^3_{n>0}$, it follows that the state should be also annihilated by $\hat{J}^3_{n\geq 0}$.

Before addressing the unitarity issue itself, we should first discuss the quantum numbers characterizing a state belonging to the supersymmetric $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset. This is the major subtlety in generalizing the result of [2].

Consider the decompositions

$$\begin{aligned} J^3_0 &= j^3_0 + \hat{J}^3_0 \\ L_0 &= L^b_0 + L^f_0 \end{aligned} \tag{3.2}$$

where L^b_0 and L^f_0 are the zero modes of

$$\begin{aligned} T^b &= \frac{1}{2k}[j^+j^- + j^-j^+] - \frac{1}{k}j^3j^3 \\ T^f &= -\frac{1}{2k}[\psi^+\partial\psi^- + \psi^-\partial\psi^+] + \frac{1}{k}\psi^3\partial\psi^3\,, \end{aligned} \tag{3.3}$$

respectively, with $T = T^b + T^f$.

Due to the direct product structure, a basis can be chosen for the $\mathrm{SL}(2,\mathbb{R})$ module made up of states diagonal in $j^3_0, \hat{J}^3_0, L^b_0$ and L^f_0. The eigenvalues of j^3_0 and $\hat{J}^3_0$ will be called t and s, and the levels of L^b_0 and L^f_0 will be denoted by N^b and N^f. Calling m and N the eigenvalue of J^3_0 and the level of L_0, (3.2) implies $m = t + s$ and $N = N^b + N^f$.

A state belonging to the coset can be assumed to have definite N, t and s quantum numbers, because the commutators

$$\begin{aligned} [L_0, J^3_n] &= -nJ^3_n \\ [L_0, \psi^3_n] &= -n\psi^3_n \\ [j^3_0, J^3_n] &= [j^3_0, \psi^3_n] = 0 \\ [\hat{J}^3_0, J^3_n] &= [\hat{J}^3_0, \psi^3_n] = 0 \end{aligned} \tag{3.4}$$

imply that $J^3_{n>0}$ and $\psi^3_{n>0}$ annihilate states with different N, t or s separately.

On the other hand, a state in the coset will not in general have definite N^b and N^f quantum numbers, because the nondiagonal commutators

$$\begin{aligned} [L^b_0, J^3_n] &= -nj^3_n \\ [L^f_0, J^3_n] &= -n\hat{J}^3_n \end{aligned} \tag{3.5}$$

show that action of J^3_n and ψ^3_n will generally mix states with different N^b and N^f. For example, rewriting the level $N = 1$ coset state (3.1) as

$$|x\rangle = j^3_{-1}|j,t\rangle + \left(\frac{1}{k} + \frac{1}{2}\right)\psi^+_{-1/2}\psi^-_{-1/2}|j,t\rangle \tag{3.6}$$

it is clear that it has $s = 0$ and $t = m$, but N^b and N^f are different for the two terms.

Let's call N^b_{max} to the maximum value of N^b appearing in the terms of the expansion of a coset state. In the same term will appear $N^f_{\mathrm{min}} = N - N^b_{\mathrm{max}}$ as the minimum value of N^f. It is clear that the structure of the algebra implies that $2N^f_{\mathrm{min}} \geq s^2$ and $t \geq j - N^b_{\mathrm{max}}$.

Let's see now that the norm of $|x\rangle$ is positive. Consider

$$\langle x|L_0|x\rangle = \left(-\frac{j(j-1)}{k} + N\right)\langle x|x\rangle \tag{3.7}$$

We will use now the explicit form (2.5) of L_0, but we write the terms that come from $j^3 j^3$ in (2.4 as coming from

$$j^3 j^3 = J^3 J^3 - \hat{J}^3 J^3 - J^3 \hat{J}^3 + \hat{J}^3 \hat{J}^3\,,$$

whose zero mode is

$$(J^3_0)^2 - 2J^3_0\hat{J}^3_0 + 2\sum_{n=1}^{\infty}(J^3_{-n}J^3_n - \hat{J}^3_{-n}J^3_n - J^3_{-n}\hat{J}^3_n) + \frac{2}{k}\sum_{q=\frac{1}{2}}^{\infty} q(\psi^+_{-q}\psi^-_q + \psi^-_{-q}\psi^+_q)\,, \tag{3.8}$$

where we have used

$$\hat{J}^3\hat{J}^3 = \frac{1}{k^2}\!:\psi^+\psi^- :: \psi^+\psi^- : = -\frac{1}{k}\psi^+\partial\psi^- - \frac{1}{k}\psi^-\partial\psi^+\,.$$

The resulting expression for L_0 is

$$L_0 = \frac{1}{k}\left[\frac{1}{2}(j_0^+ j_0^- + j_0^- j_0^+) + \sum_{m=1}^{\infty} j_{-m}^+ j_m^- + j_{-m}^- j_m^+\right] + \\ + \frac{1}{k}\left[(J_0^3)^2 - 2J_0^3 \hat{J}_0^3 + 2\sum_{n=1}^{\infty}(J_{-n}^3 J_n^3 - \hat{J}_{-n}^3 J_n^3 - J_{-n}^3 \hat{J}_n^3)\right] + \\ + \frac{1}{k}\sum_{m=\frac{1}{2}}^{\infty} m\left[\left(1-\frac{2}{k}\right)(\psi_{-m}^+ \psi_m^- + \psi_{-m}^- \psi_m^+) - 2\psi_{-m}^3 \psi_m^3\right] \tag{3.9}$$

Inserting this expression in (3.7) and annihilating the modes $J_{n>0}^3, \psi_{n>0}^3$ ($J_{n<0}^3, \psi_{n<0}^3$) against $|x\rangle$ ($\langle x|$), (3.7) can be rearranged into

$$\langle x|x\rangle = \frac{\langle x|[\sum_{p=0}^{\infty} j_{-p}^+ j_p^- + \sum_{p=1}^{\infty} j_{-p}^- j_p^+ + \left(1-\frac{2}{k}\right)\sum_{q=\frac{1}{2}}^{\infty} q(\psi_{-q}^+ \psi_q^- + \psi_{-q}^- \psi_q^+)]|x\rangle}{t(t-1) - j(j-1) + kN - s^2}\,. \tag{3.10}$$

We will see that $\langle x|x\rangle$ is positive by showing that both the numerator and the denominator of (3.10) are positive.

Let's consider first the denominator. When $t \geq j$ it can be written as

$$t(t-1) - j(j-1) + kN_{\max}^b + (2N_{\min}^f - s^2) + (k-2)N_{\min}^f \tag{3.11}$$

which is positive for $k > 2$. When $t < j$, we rewrite it as

$$k(N_{\max}^b + t - j) + (j-t)(j-t-1) + 2\left(\frac{k}{2} + 1 - j\right)(j-t) + (2N_{\min}^f - s^2) + (k-2)N_{\min}^f \tag{3.12}$$

which is also positive for $k > 2$ and $j < k/2 + 1$. So the denominator is always positive.

Regarding the numerator, we proceed by induction on N and m. We assume that unitarity holds for states in the coset module with level lower than N, or J_0^3 eigenvalue lower than m.

Note that the numerator is the sum of norms of states of the form $j_{p>0}^{\pm}|x\rangle, \psi_{q>0}^{\pm}|x\rangle$ all of which have level lower than N, and $j_0^-|x\rangle$ which has level N but lower J_0^3 eigenvalue. But the norm of these states is not guaranteed to be positive by the induction hypothesis, since, although they are still annihilated by $\psi_{r>0}^3$, they are not necessarily annihilated by $J_{n>0}^3$.

Now, let $\mathbf{P}$ be the projection operator on highest weight states of J^3. Since every state of the theory can be obtained by applying lowering operators $J_{n<0}^3$ to highest weight states, we have the completeness relation

$$1 = \mathbf{P} + \sum_{n>0}\left(-\frac{2}{kn}\right) J_{-n}^3 \mathbf{P} J_n^3 + \\ + \frac{1}{2!}\sum_{n_1,n_2>0}\left(-\frac{2}{kn_1}\right)\left(-\frac{2}{kn_2}\right) J_{-n_1}^3 J_{-n_2}^3 \mathbf{P} J_{n_2}^3 J_{n_1}^3 + \cdots\,. \tag{3.13}$$

Inserting this expression in norms like $\langle x|j^+_{-p}j^-_p|x\rangle$ and $\langle x|\psi^+_{-p}\psi^-_p|x\rangle$ infinite series are obtained, typical terms being

$$\frac{1}{d!}\left(-\frac{2}{k}\right)^d \frac{1}{n_1\ldots n_d}\langle x|j^+_{-p}J^3_{-n_1}\cdots J^3_{-n_d}\mathbf{P}J^3_{n_d}\cdots J^3_{n_1}j^-_p|x\rangle \tag{3.14}$$

$$\frac{1}{d!}\left(-\frac{2}{k}\right)^d \frac{1}{n_1\ldots n_d}\langle x|\psi^+_{-q}J^3_{-n_1}\cdots J^3_{-n_d}\mathbf{P}J^3_{n_d}\cdots J^3_{n_1}\psi^-_q|x\rangle\,. \tag{3.15}$$

Commuting the J^3_n and annihilating them against $|x\rangle$ and $\langle x|$, the above expressions can be brought to

$$\frac{1}{d!}\left(-\frac{2}{k}\right)^d \frac{1}{n_1\cdots n_d}\langle x|j^+_{-(p+n_1+\cdots+n_d)}\mathbf{P}j^-_{(p+n_1+\cdots+n_d)}|x\rangle \tag{3.16}$$

$$\frac{1}{d!}\left(-\frac{2}{k}\right)^d \frac{1}{n_1\cdots n_d}\langle x|\psi^+_{-(q+n_1+\cdots+n_d)}\mathbf{P}\psi^-_{(q+n_1+\cdots+n_d)}|x\rangle\,. \tag{3.17}$$

Note that the $\langle x|\cdots|x\rangle$ factors are positive by the induction hypothesis. Using this method, the numerator of (3.10) can be put into the form

$$\langle x|\left[\sum_{p=0}^{\infty}F_p(y)j^+_{-p}\mathbf{P}j^-_p+\sum_{p=1}^{\infty}F_{p-1}(y)j^-_{-p}\mathbf{P}j^+_p+(1-y)\sum_{q=\frac{1}{2}}^{\infty}H_q(y)(\psi^+_{-q}\mathbf{P}\psi^-_q+\psi^-_{-q}\mathbf{P}\psi^+_q)\right]|x\rangle \tag{3.18}$$

where $y=2/k$ and

$$\begin{aligned} F_p(y) &= \sum_{r=0}^{p} f_r(y)\,,\\ H_q(y) &= \sum_{r=0}^{q-1/2}(q-r)f_r(y) \end{aligned} \tag{3.19}$$

where $f_0(y)=1$ and

$$f_{r\geq 1}(y)=\sum_{d=1}^{\infty}\frac{(-y)^d}{d!}\sum_{n_i>0}\frac{1}{n_1\cdots n_d}\delta_{n_1+n_2+\cdots n_d,r}\,. \tag{3.20}$$

It is clear that (3.18) will be positive if $F_p(y)$ and $H_q(y)$ are positive and $y<1$. In order to express $F_p(y)$ and $H_q(y)$ in a closed form, we first consider, following [2], the generating function for $f_r(y)$

$$\begin{aligned} f(y,z) &= \sum_{r=0}^{\infty}f_r(y)z^r\\ &= 1+\sum_{d=1}^{\infty}\frac{(-y)^d}{d!}\prod_{i=1}^{d}\left(\sum_{n_i=1}^{\infty}\frac{z^{n_i}}{n_i}\right)\\ &= \sum_{d=0}^{\infty}\frac{(-y)^d}{d!}\left[-\ln(1-z)\right]^d\\ &= (1-z)^y\,. \end{aligned} \tag{3.21}$$

Now $f_r(y)$ can be obtained by derivation to find

$$\begin{aligned} f_1(y) &= -y\,, \\ f_{r>1}(y) &= -\frac{y}{r}(1-y)\left(1-\frac{y}{2}\right)\cdots\left(1-\frac{y}{r-1}\right) \end{aligned} \tag{3.22}$$

and from (3.19) we obtain $F_0(y) = 1$ and

$$F_{p\geq 1}(y) = (1-y)\left(1-\frac{y}{2}\right)\cdots\left(1-\frac{y}{p}\right) \tag{3.23}$$

which is positive for $y < 1$. Regarding $H_q(y)$ we see from (3.19) and (3.22) that

$$\begin{aligned} H_{\frac{1}{2}}(y) &= 1/2 \\ H_{\frac{3}{2}}(y) &= \frac{3}{2}F_1(y) + y \\ H_{q>\frac{3}{2}}(y) &= qF_{q-\frac{1}{2}} + y + y(1-y) + \cdots + y(1-y)\left(1-\frac{y}{2}\right)\cdots\left(1-\frac{y}{q-3/2}\right) \end{aligned} \tag{3.24}$$

so that $H_q(y)$ is also positive for $y < 1$. It follows then that (3.18) is positive for $y < 1$, that is, for $k > 2$.

4. No-ghost theorem for fermionic strings in the discrete series of the NS sector

The no-ghost theorem guarantees that states satisfying the physicality conditions (2.8) have positive norm. For the fermionic string, it was shown in [3] that physical states belong to the supersymmetric $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset, modulo spurious states. To this statement the result of section 3 must be added. For the spectrally flowed representations of $\widehat{\mathrm{SL}}(2,\mathbb{R})$ a proof was given in [4] for the bosonic case.

In section 4.1, for the sake of completeness, we will review the proof of [3] for unflowed fermionic strings, and then generalize it to the flowed case in section 4.2. We will work out the details of the $\mathcal{D}_j^+$ representations, the $\mathcal{D}_j^-$ case being similar.

4.1 Unflowed states

We define the subspace $\mathcal{F}$ of the Hilbert space as consisting of states $|f\rangle$ which are primaries of $J^3(z), \psi^3(z), T(z)$ and $G(z)$, that is,

$$J_n^3|f\rangle = \psi_n^3|f\rangle = L_n|f\rangle = G_n|f\rangle = 0, \qquad n > 0\,. \tag{4.1}$$

Notice that by the result of section 3, states in $\mathcal{F}$ have positive norm. The proof has three steps.

Step 1: If $|N,m,\nu\rangle$ is an orthogonal basis of $\mathcal{F}$ at level N and J_0^3 eigenvalue m, the states of the form

$$G^{\varepsilon_0}_{-1/2}\cdots G^{\varepsilon_a}_{-a-1/2}\, L^{\lambda_1}_{-1}\cdots L^{\lambda_d}_{-d}$$
$$(\psi^3_{-1/2})^{\delta_0}\cdots(\psi^3_{-b-1/2})^{\delta_b}\,(J^3_{-1})^{\mu_1}\cdots(J^3_{-n})^{\mu_n}|N,m,\nu\rangle\,, \tag{4.2}$$

with m fixed and ν varying, are linearly independent and form a basis of the Hilbert space for states with level $M\geq N$ and J_0^3 eigenvalue m.
Proof. Defining

$$\begin{aligned} T^3 &= -\frac{1}{k}J^3J^3-\frac{1}{k}\psi^3\partial\psi^3 \\ G^3 &= -\frac{2}{k}J^3\psi^3, \end{aligned} \tag{4.3}$$

the fields J^3,ψ^3,T^3 and G^3 form a $c=3/2$ supersymmetric timelike U(1) algebra. We can further define

$$\begin{aligned} L^c_n &= L_n-L^3_n \\ G^c_n &= G_n-G^3_n\,, \end{aligned} \tag{4.4}$$

which, by construction, commute with ψ^3_n and J^3_n. Moreover, L^c_n and G^c_n form a $\hat{c}=9$ ($c=27/2$) supersymmetric Virasoro algebra. It is clear from (4.1), (4.3) and (4.4), that $|N,m,\nu\rangle$ is also a primary of L^c_n, its weight with respect to ${L^c}_0$ being,

$$h^c=-\frac{j(j-1)}{k}+N+\frac{m^2}{k}+h\,, \tag{4.5}$$

where h, is the highest weight of the internal unitary $\mathcal{N}$ CFT.

Using (4.4) we can put the states (4.2) in one-to-one correspondence with states

$$(G^c_{-1/2})^{\varepsilon_0}\cdots(G^c_{-a-1/2})^{\varepsilon_a}\,(L^c_{-1})^{\lambda_1}\cdots(L^c_{-d})^{\lambda_d}$$
$$(\psi^3_{-1/2})^{\delta_0}\cdots(\psi^3_{-b-1/2})^{\delta_b}\,(J^3_{-1})^{\mu_1}\cdots(J^3_{-n})^{\mu_n}|N,m,\nu\rangle\,, \tag{4.6}$$

We want to show that for each m, the states (4.6) are a basis for the whole Hilbert space. Linear independence of the states (4.6) is equivalent to the non-singularity of the determinant of their inner products. Since states built upon different $|N,m,\nu\rangle$'s or with different (ψ^3_n,J^3_n) content are orthogonal, the determinant factorizes into Kac determinants of the L^c_n,G^c_n supersymmetric Virasoro algebra with highest weight h^c and $\hat{c}=9$. These determinants in the NS sector are singular only for $h^c\leq 0$ [24], so we need $h^c>0$. Since the structure of the algebra implies that $m\geq j-N-1/2$, rewriting (4.5) as

$$h^c=\frac{(m-j)^2}{k}+\frac{2j}{k}\left(N+m+\frac{1}{2}-j\right)+\frac{2N}{k}\left(\frac{k}{2}-j\right)+h\,, \tag{4.7}$$

we see that it is strictly positive if $j<k/2$. Rewriting further (4.5) as

$$h^c=\frac{1}{k}\left[\left(j-\frac{k}{2}\right)\left(\frac{k}{2}+1-j\right)+\left(m-\frac{k}{2}\right)^2\right]+\left(N-j+m+\frac{1}{2}\right)+h \tag{4.8}$$

we see that h^c is also strictly positive for $k/2\leq j<k/2+1$, and linear independence follows.

In order to see that the states (4.6) generate the whole Hilbert space, let's define as $\mathcal{H}^{(M)}$ the subspace of states with L_0 level M. For the states (4.6),

$$M = N + \sum_{s=0}^{a}(1/2+s)\varepsilon_s + \sum_{s=1}^{m}\lambda_s s + \sum_{s=0}^{b}\left(\frac{1}{2}+s\right)\delta_s + \sum_{s=1}^{n}\mu_s s\,.$$

Let's also define as $\mathcal{G}^{(M)}$ the subspace of $\mathcal{H}^{(M)}$ generated by states (4.6) such that $M > N$. We proceed now by induction on M as in [3, 21]. For $M = 0$, $\mathcal{H}^{(M)}$ is just formed by the representation $\mathcal{D}_j^+$ of the zero modes.

Let's assume the induction hypothesis for states with level lower than M. The linear independence argument above has shown that there are no null states among the descendants (4.6) of states in $\mathcal{F}$, that is, no states in $\mathcal{G}^{(M)}$ which are orthogonal to all states in $\mathcal{G}^{(M)}$. It follows that $\mathcal{H}^{(M)}$ is the direct sum of $\mathcal{G}^{(M)}$ and its orthogonal complement.

Consider a state in the orthogonal complement of $\mathcal{G}^{(M)}$. From the induction hypothesis that state is annihilated by $L_{n>0}, G_{n>0}, \psi^3_{n>0}$ and $J^3_{n>0}$, that is, it belongs to $\mathcal{F}$. It follows that $\mathcal{H}^{(M)} = \mathcal{G}^{(M)} \oplus \mathcal{F}^{(M)}$ where $\mathcal{F}^{(M)}$ are states in $\mathcal{F}$ at level M, i.e., the states (4.6) form a complete basis at each level M and each J_0^3 eigenvalue m.

Step 2: A physical state can be decomposed into the sum of two physical states, one of them spurious and the other not.
Proof. A state is called spurious if it is a linear combination of states (4.2) with $\varepsilon_i \neq 0$ or $\lambda_i \neq 0$. Given a physical state expressed in the basis (4.2), it can be written as a state with no L_n, G_n plus a spurious state. It can be checked that the action of $L_{n>0}, G_{n>0}$ on a spurious state leaves it spurious provided $L_0 = 1/2$ and $c = 15$ [21]. Moreover, the action of $L_{n>0}, G_{n>0}$ on states without L_{-n}, G_{-n} won't produce new L_{-n}, G_{-n}. Since (4.2) is a basis, it follows that both the spurious and the nonspurious states satisfy the physicality conditions (2.8) separately.

Note that since the inner product of a physical and a spurious state vanishes we should only check unitarity for the state with no L_{-n}, G_{-n}.

Step 3: A physical state $|x\rangle$ involving neither L_{-n} nor G_{-n} (when written as (4.2)), belongs to $\mathcal{F}$.
Proof. Using the decomposition (4.4) it is clear that $|x\rangle$ is annihilated by $L^3_{n>0}$ and $G^3_{n>0}$. A sufficient condition under which $|x\rangle$ will also be annihilated by $J^3_{n>0}$, $\psi^3_{n>0}$ is that $m \neq 0$.

The reason is that when $m \neq 0$ the Hilbert space of L^3_{-n}, G^3_{-n} acting on $\mathcal{F}$ has no null descendants, because it has a negative highest weight $-m^2/2k$, and null descendants of a $c = 3/2$ NS supersymmetric Virasoro algebra only appear for nonnegative highest weights [24].

Assuming the absence of null Virasoro descendants, the L^3_{-n}, G^3_{-n} Hilbert space and that generated by J^3_{-n}, ψ^3_{-n} will be the same.[2] So our $|x\rangle$ state, which is a J^3_{-n}, ψ^3_{-n} descendant, can be expressed as a L^3_{-n}, G^3_{-n} descendant. But being annihilated by $L^3_{n>0}$ and $G^3_{n>0}$, $|x\rangle$ can only be the highest weight state, which is also annihilated by $J^3_{n>0}$, $\psi^3_{n>0}$.

[2] This follows by simply counting the number of states at each level.

The above argument does not hold for states with $m = 0$, and we treat this case independently. Let's consider the appearance of physical states with $m = 0$ for different values of j. Remember that the mass shell condition is

$$-\frac{j(j-1)}{k} + N + h - \frac{1}{2} = 0 \tag{4.9}$$

For $0 < j < 1$ there are no states with $m = 0$ because m differs from j by integer values.

For $j = 1$, eq. (4.9) implies that $N = 0$ or $N = 1/2$. In the first case there are no states in $\mathcal{D}_j^+$ with $m < j$. In the second case, the only state is

$$\psi^-_{-\frac{1}{2}}|j = 1, m = 1\rangle \tag{4.10}$$

which is physical and belongs to $\mathcal{F}$.

For $j > 1$, since for $m = 0$ we have $N + 1/2 \geq j$, the left hand side of (4.9) is

$$-\frac{j(j-1)}{k} + N + h - \frac{1}{2} \geq -\frac{j(j-1)}{k} + \frac{k(j-1)}{k} + h = \frac{(k-j)(j-1)}{k} + h > 0 \tag{4.11}$$

because $j < \frac{k}{2} + 1$ implies $(k - j) > \frac{(k-2)}{2}$, which is positive for $k > 2$. Thus (4.9) cannot be satisfied either.

It follows that every physical state is annihilated by $J^3_{n>0}, \psi^3_{n>0}$ and its norm is positive by the result of section 3.

4.2 Flowed states

In this case, the states are built by the action of $\tilde{\psi}^{3,\pm}_{-n}$ and $\tilde{J}^{3,\pm}_{-n}$ on $|\tilde{j}, \tilde{t}\rangle$, but the physicality conditions (2.8) are still imposed with the unflowed operators L_n and G_n. Since we are considering the $\mathcal{D}_j^+$ series, we need only consider spectral flow with $w > 0$.

Step 1: By the same arguments as before, we know that the states as those in (4.2), but written with $\tilde{\psi}^3_{-n}, \tilde{J}^3_{-n}, \tilde{L}_{-n}, \tilde{G}_{-n}$ form[3] a basis for states built upon $|\tilde{j}, \tilde{t}\rangle$. But replacing in (4.2) these operators with

$$\begin{aligned}
\tilde{J}^3_n &= J^3_n \\
\tilde{\psi}^3_n &= \psi^3_n \\
\tilde{L}_n &= L_n + wJ^3_n \\
\tilde{G}_n &= G_n + w\psi^3_n
\end{aligned} \tag{4.12}$$

for $n < 0$, they can be put in one to one correspondence with the states (4.2). In other words, the states (4.2), with $\psi^3_{-n}, J^3_{-n}, L_{-n}, G_{-n}$, are a basis for the Hilbert space built by $\tilde{\psi}^{3,\pm}_n, \tilde{j}^{3,\pm}_n$ upon $|\tilde{j}, \tilde{t}\rangle$.

Step 2: It is the same.

[3]Note that the conditions (4.1) defining $\mathcal{F}$ can equivalently be imposed with either flowed or unflowed operators.

Step 3: The physicality conditions (2.8) for $|x\rangle$ again imply that $L^3_{n>0}$ and $G^3_{n>0}$ annihilate it, and this in turn means that also $\tilde{J}^3_{n>0}, \tilde{\psi}^3_{n>0}$ annihilate $|x\rangle$ provided $m = \tilde{m} + \frac{kw}{2} \neq 0$. Let's see that no states with $m = 0$ can appear in the flowed representations by using the flowed mass shell condition (2.9),

$$-\frac{\tilde{j}(\tilde{j}-1)}{k} - wm + \frac{kw^2}{4} + \tilde{N} + h - \frac{1}{2} = 0 \tag{4.13}$$

In the following we use that $\tilde{j} < k/2 + 1$ implies that $-\frac{\tilde{j}(\tilde{j}-1)}{k} > -k/4 - 1/2$.

For $w = 1$ the left hand side of (4.13) for $m = 0$ is

$$-\frac{\tilde{j}(\tilde{j}-1)}{k} + \frac{k}{4} + \tilde{N} + h - \frac{1}{2} > \tilde{N} + h - \frac{1}{2} \tag{4.14}$$

and using $\tilde{m} = -k/2$ the right hand side of (4.14) is

$$\left(\tilde{N} - \tilde{j} + \tilde{m} + \frac{1}{2}\right) + \tilde{j} - \tilde{m} - 1 + h = \left(\tilde{N} - \tilde{j} + \tilde{m} + \frac{1}{2}\right) + \tilde{j} + \frac{(k-2)}{2} + h > 0 \tag{4.15}$$

because $(\tilde{N} - \tilde{j} + \tilde{m} + 1/2) \geq 0$ and $k > 2$, so that (4.13) is not satisfied.

For $w \geq 2$, the left hand side of (4.13) for $m = 0$ is

$$-\frac{\tilde{j}(\tilde{j}-1)}{k} + \frac{kw^2}{4} + \tilde{N} + h - \frac{1}{2} > \frac{k}{4}(w^2 - 1) - 1 + \tilde{N} + h > \frac{1}{2} + \tilde{N} + h > 0 \tag{4.16}$$

because $k > 2$, so that (4.13) is not satisfied either.

Having seen that physical states are annihilated by $\tilde{J}^3_{n>0}, \tilde{\psi}^3_{n>0}$, unitarity follows again from the result of section 3.

5. Conclusions

We have proved the no-ghost theorem for the NS sector of discrete representations of fermionic strings in $AdS_3 \times \mathcal{N}$. The result is relevant for a whole family of vacua, such as those yielding superconformal supersymmetry in the boundary CFT theory [25, 26].

For flowed fermionic strings in $\mathcal{C}_j$ representations, the result of section 4.2 can be easily generalized, but then we should prove the supersymmetric coset unitarity in the $\mathcal{C}_j$ sector.

The proof of the no-ghost theorem discussed above is a curved space generalization of the proof for flat space in the old covariant quantization scheme [20, 21]. On the other hand, both for $\mathcal{D}^{\pm}_j$ and for $\mathcal{C}_j$ representations, the unitarity requirement for the supersymmetric coset can be bypassed in the proof of the no-ghost theorem by using instead a BRST quantization scheme. In that case one can rely upon the bosonic coset unitarity by using the decomposition into j^a and ψ^a currents [27]. In any case, the unitarity of the supersymmetric coset is relevant by itself, due to the wealth of models based on it.

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